Final exam for Quantum Physics 1 - 2014-2015

Thursday 30 October 2014, 9:00 - 12:00

<u>READ THIS FIRST</u>:

- The course is in English, please answer in English as much as you can.
- Clearly write your name and study number on each answer sheet that you use.
- Start each question (number 1, 2, etc.) on a new answer sheet.
- On the first answer sheet, write the total number of answer sheets that you turn in.
- When turning in your answers, please stack your answer sheets in the proper order, and **staple** them together (stapler is at desk exam supervisors).
- The exam has several questions, it continues on the backsides of the papers.
- Note that the lower half of this page lists some useful formulas and constants.
- The exam is open book with limits. You are allowed to use the book by Griffiths, the handouts *Extra note on two-level systems and exchange degeneracy for identical particles* and *Feynman Lectures chapter III-1*, and one A4 sheet with your own notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- If you are ready with the exam, please fill in the **course-evaluation question sheet**. You can keep working on the exam until the scheduled end time, and fill it in shortly after that if you like.
- The full exam is 80 points (4 problems of 20 points).

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs
Energy units		1 eV $= 1.6 \cdot 10^{-19}$ J

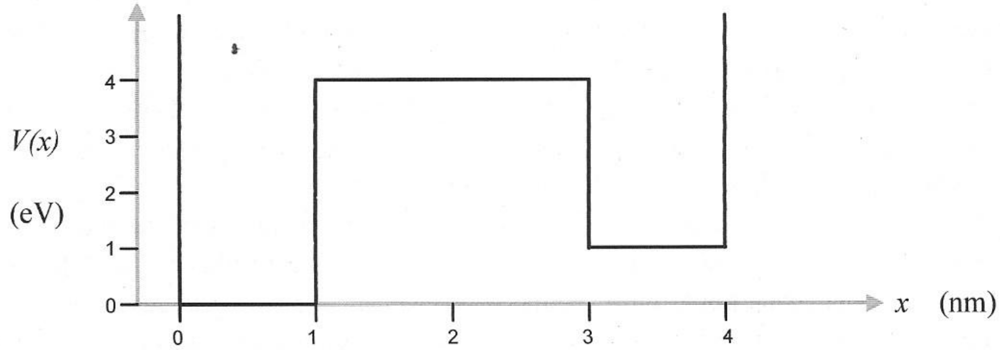
Fourier relations between x-representation and k-representation of a state

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, \Psi(x)\, dx$$

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \overline{\Psi}(k)\, dk$$

Problem 1 *Hint for good notation in this problem:*
during your calculations use x_1 for $x=1$ nm, V_4 for $V(x) = 4$ eV, etc.

a) [5 points]
Consider a quantum particle in the following one-dimensional quantum well (see figure, sketch as a function of position x).

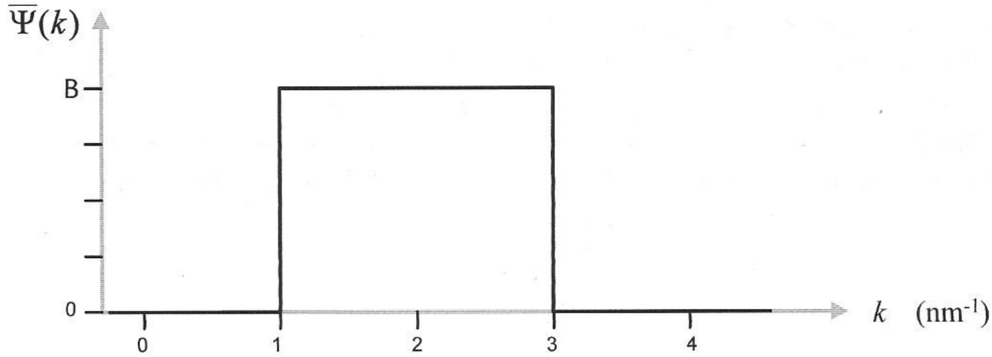


The mass of the particle is $1 \cdot 10^{-20}$ kg. With a high-energy beam, the particle is prepared in the following state:

$$\Psi(x) = \begin{cases} iAx\,, & 0\text{ nm} < x < 4\text{ nm} \\ 0\,, & \text{all other } x \end{cases},$$

where A is a real constant, and $A > 0$ and $i = \sqrt{-1}$. What is the expectation value for potential energy for the particle in this state? Your final answer must be a number.

b) [5 points]
Consider a different quantum particle. It is an electron that behaves as a free particle (it feels a potential energy $V = 0$ everywhere). At time $t = 0$ it is prepared in a state as in the following graph (which describes the state in k-representation, the wave function in the graph is real).



Calculate for the particle in this state the expectation value for its kinetic energy (with respect to the particle at rest, where it has zero energy). Your final answer must be a number.

c) [5 points]
For the particle in the state of question b), determine the expectation value for position at time $t = 0$.
Hint: During your calculation, work towards an expression in the form of a sinc function.

d) [5 points]
For the particle in the state of question b), make a rough estimate of how much time it will take for the quantum uncertainty in position (Δx) to grow from the value it has at $t = 0$ to a value that is a factor 100 larger.

Problem 2 *In this problem you might use*

$$\int_0^1 x\sin^2(n\pi x)\,dx = \frac{1}{4}, \quad \text{for } n \text{ integer and } n > 0,$$

$$\int_0^1 x\sin(\pi x)\sin(2\pi x)\,dx = -\frac{8}{9\pi^2}.$$

Consider the following quantum system. It has a single quantum particle in a one-dimensional particle-in-a-box system, where the potential for the particle outside the box is infinite, and inside the box the potential $V = 0$. The position of the electron is described by a coordinate x. The width of the box is L, with the walls at $x = 0$ and $x = L$. The mass of the particle is m. First assume that the system is very well isolated from its environment.

a) [1 point]
Valid descriptions of the ground state and first excited state of this system are (in x-representation)

$$\varphi_g(x) = \sqrt{\tfrac{2}{L}}\sin(\tfrac{\pi}{L}x),$$

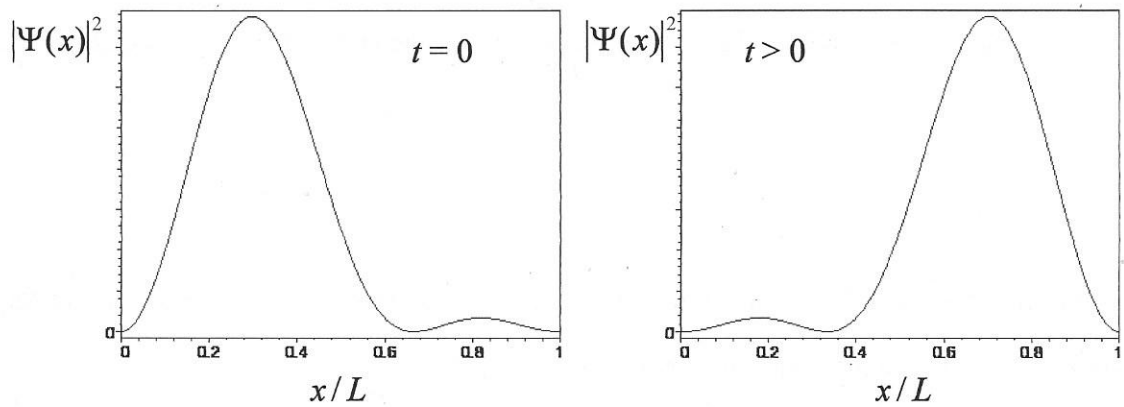
$$\varphi_e(x) = \sqrt{\tfrac{2}{L}}\sin(\tfrac{2\pi}{L}x).$$

Give the energy of the ground state (call it E_g) and the first excited state (call it E_e), with respect to $V = 0$, in terms of L, m and natural constants (you can just write them, there is no need to show a derivation).

b) [6 points]
At some point in time defined as $t = 0$, the system is in a quantum superposition of φ_g and φ_e,

$$\Psi(x, t = 0) = \tfrac{1}{\sqrt{2}}\left(\varphi_g(x) + \varphi_e(x)\right).$$

Since this state is not an energy eigenstate it will change as a function of time. The figure below here shows the probability density associated with $\Psi(x,t)$ for $t = 0$ and for some time $t > 0$. Note that the two graphs are each other's mirror image with respect to $x/L = ½$.



Calculate how long after $t = 0$ the probability density of $\Psi(x,t)$ has evolved into the form as in the right graph (for the first time after $t = 0$). Express your answer in the constants that were mentioned above here.

Hint 1: First try to write down a state that agrees with the right graph, in the form of a superposition of φ_g and φ_e.

Hint 2: Note that a global phase factor (also called arbitrary phase factor) for the full state has no influence on the shape of the probability density.

c) [7 points]
For the system in the state of question b) at $t = 0$, calculate how the expectation value for position depends on time for $t > 0$. Work it out as far as possible, such that you can clearly explain what the dynamics is of the system in this state.

d) [6 points]
Now assume that the system is weakly coupled to its environment. This environment is very dark and cold ($k_B T << E_e$-E_g, where T the temperature and k_B the Boltzmann constant). For the system in the state of question b) at $t = 0$, make a sketch of how the expectation value for position depends on time for $t > 0$ for this case where the system is weakly coupled to its environment. Explain your answer.

Problem 3
Note on notation: the book uses the notation σ_A instead of the notation ΔA that is used in this question.

For spin-1/2 systems, the operators for $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are often represented as 2 by 2 matrices, in a basis that uses the spin-up and spin-down states along z-direction as basis vectors (these two states are denoted as $|\uparrow\rangle$ and $|\downarrow\rangle$).

For this problem, assume that such a spin is in a state

$$|\Psi_1\rangle = a|\uparrow\rangle + b|\downarrow\rangle, \quad \text{with} \quad a \geq 0, \; b \geq 0, \; \text{and } a, b \text{ both real.}$$

Note that you should for this problem only consider a and b values that agree with the above.

a) [2 points]

Prove that for the all the a and b values that are possible for the state $|\Psi_1\rangle$ the expectation value for $\hat{S}_y$ is 0 $\hbar$.

b) [5 points]

For the spin in a state as $|\Psi_1\rangle$, analyze how the quantum uncertainty in S_z depends on a in the following manner. For short notation, write p_a for a^2.

Make a graph of $(\Delta S_z)^2$ as a function of p_a, for the range $p_a = 0$ to $p_a = 1$.

c) [5 points]

For the spin in a state as $|\Psi_1\rangle$, also make a graph of $(\Delta S_x)^2$ as a function of p_a, for the range $p_a = 0$ to $p_a = 1$.

d) [4 points]

If you answered c) correctly, your graph has at least one point where $(\Delta S_x)^2 = 0\,\hbar^2$. Describe for which state(s) $|\Psi_1\rangle$ this holds (that is, mention the a and b values). Explain what is going on at this point (these points).

e) [4 points]

There exists an uncertainty relation between ΔS_x and ΔS_z (such a relation is analogous to the Heisenberg uncertainty relation $\Delta x \cdot \Delta p \geq \hbar / 2$ for position and momentum of a particle). If you have worked out b) and c) correctly, you will notice that for some values of p_a the value of $\Delta S_x \cdot \Delta S_z = 0\,\hbar^2$.

Use the generalized uncertainty principle for explaining that this is allowed when the spin is in a state as $|\Psi_1\rangle$.

Problem 4

a) [5 points]

A system is composed of two particles which are both in a state that has angular momentum (with angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$, respectively).

The quantum numbers for the length of the angular momentum vector of particle 1 and particle 2 are $l_1 = 9/2$ and $l_2 = 5/2$. There are no other angular momentum contributions to the total angular momentum of this system.

An apparatus is going to measure the length of the total angular momentum vector $\vec{L}_{TOT} = \vec{L}_1 + \vec{L}_2$. What measurement outcomes are possible?

b) [5 points]

Assume that we just did the measurement of question a), and that it gave as outcome the lowest value of all possible outcomes. In a next measurement, the apparatus is going to measure the *y*-component of the total angular momentum vector. What are now the possible measurement outcomes?

Note: if you have no answer for question a), you can still try to answer this question by assuming some answer for a). If you do this, clearly mark ASSUME with your answer.

c) [5 points] **(also read question d) now)**

A certain atom is in a state with its total orbital angular momentum vector $\vec{L}$ (described by the operator $\hat{L}$) defined by orbital quantum number $l = 1$.

For the system in this state, the operator for the *z*-component of angular momentum is $\hat{L}_z$. This operator can be represented as a matrix, and the ket-states as column vectors, while using the states $|+_z\rangle$, $|0_z\rangle$ and $|-_z\rangle$ as basis states, according to

$$\hat{L}_z \leftrightarrow \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad |+_z\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \quad |-_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using this same basis for the representation, the operator and eigenstates for the system's *x*-component of angular momentum are given by

$$\hat{L}_x \leftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad |+_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad |0_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \text{ and } \quad |-_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}.$$

At some point the system is in the normalized state $|\Psi_2\rangle = \sqrt{\frac{1}{5}}\,|+_z\rangle + \sqrt{0}\,|0_z\rangle + \sqrt{\frac{4}{5}}\,|-_z\rangle$, and you are going to measure the *x*-component of the system's angular momentum. What are the possible measurement outcomes? Explain your answer.

d) [5 points]

What is the probability for each of the possible measurement outcomes of question c), when the system is in state $|\Psi_2\rangle$ before the measurement?